# Introduction

Paradoxes are fun. In most cases, they are easy to state and immediately provoke one into trying to "solve" them.

One of the hardest paradoxes to handle is also one of the easiest to state: the Liar paradox. One version of it asks you to consider the man who simply says, "What I am now saying is false." Is what he says true or false? The problem is that if he speaks truly, he is truly saying that what he says is false, so he is speaking falsely; but if he is speaking falsely, then, since this is just what he says he is doing, he must be speaking truly. So if what he says is false, it is true; and if it is true, it is false. This paradox is said to have "tormented many ancient logicians and caused the premature death of at least one of them, Philetas of Cos." Fun can go too far.

Paradoxes are serious. Unlike party puzzles and teasers, which are also fun, paradoxes raise serious problems. Historically, they are associated with crises in thought and with revolutionary advances. To grapple with them is not merely to engage in an intellectual game, but is to come to grips with key issues. In this book, I report some famous (and some less famous) paradoxes and indicate how one might respond to them. These responses lead into some rather deep waters.

This is what I understand by a paradox: an apparently unacceptable conclusion derived by apparently acceptable reasoning from apparently acceptable premises. Appearances have to deceive, since the acceptable cannot lead by acceptable steps to the unacceptable. So, generally, we have a choice: either the conclusion is not really unacceptable, or else the starting point, or the reasoning, has some non-obvious flaw.

Paradoxes come in degrees, depending on how well appearance camouflages reality. Let us pretend that we can represent how paradoxical something is on a ten-point scale. The weak or shallow end we shall label 1; the cataclysmic end, home of paradoxes that send seismic shudders through a wide region of thought, we shall label 10. Serving as a marker for the point labeled 1 is the so-called Barber paradox: in a certain remote Sicilian village, approached by a long ascent up a precipitous mountain road, the barber shaves all and only those villagers who do not shave themselves. Who shaves the barber? If he himself does, then he does not (since he shaves *only* those who do not shave themselves); if he does not, then he indeed does (since he shaves *all* those who do not shave themselves). The unacceptable supposition is that there is such a barber – one who shaves himself if and only if he does not. The story may have sounded acceptable: it turned our minds, agreeably enough, to the mountains of inland Sicily. However, once we see what the consequences are, we realize that the story cannot be true: there cannot be such a barber, or such a village. The story is unacceptable. This is not a very deep paradox because the unacceptability is very thinly disguised by the mountains and the remoteness.

At the other end of the scale, the point labeled 10, I shall place the Liar. This placing seems the least that is owed to the memory of Philetas.

The deeper the paradox, the more controversial is the question of how one should respond to it. Almost all the paradoxes I discuss in the ensuing chapters score 6 or higher on the scale, so they are really serious. (Some of those in chapter 2 and in appendix I might be argued to rate a lower score.) This means that there is severe and unresolved disagreement about how one should deal with them. In many cases, though certainly not all (not, for example, in the case of the Liar), I have a definite view; but I must emphasize that, although I naturally think my own view is correct, other and greater men have held views that are diametrically opposed. To get a feel for how controversial some of the issues are, I suggest examining the suggestions for further reading at the ends of chapters.

Some paradoxes collect naturally into groups by subject matter. The paradoxes of Zeno which I discuss form a group because they all deal with space, time, and infinity. The paradoxes of chapter 4 form a group because they bear upon the notion of rational action. Some groupings are controversial. For example, Russell grouped the paradox about classes with the Liar paradox. In the 1920s, Ramsey argued that this grouping disguised a major difference. More recently, it has been argued that Russell was closer to the truth than Ramsey.

I have compared some of the paradoxes treated within a single chapter, but I have made no attempt to portray larger patterns. However, it is arguable that there are such patterns, or even that the many paradoxes are the many signs of one "master cognitive flaw." This last claim has been ingeniously argued by Roy Sorensen (1988).

Questions can be found in boxes throughout the text. I hope that considering these will give pleasure and will prompt the reader to elaborate some of the themes in the text. Asterisked questions are referred to in appendix II, where I have made a point that might be relevant to an answer.

I feel that chapter 6 is the hardest; it might well be left until last. The first and second are probably the easiest. The order of the others is arbitrary. Chapter 7 does not introduce a paradox, but rather examines the assumption, made in the earlier chapters, that all contradictions are unacceptable. I think it would not make much sense to one completely unfamiliar with the topics discussed in chapter 6.

I face a dilemma: I find a book disappointing if the author does not express his own beliefs. What holds him back from stating, and arguing for, the truth as he sees it? I could not bring myself to exercise this restraint. On the other hand, I certainly would not want anyone to believe what I say without first carefully considering the alternatives. So I must offer somewhat paradoxical advice: be very skeptical about the proposed "solutions"; they are, I believe, correct.

## Suggested reading

There are now a number of excellent books that deal with a spectrum of paradoxes, in particular Nicholas Rescher (2001) *Paradoxes: Their Roots, Range and Resolution* and Roy Sorensen (2003) *A Brief History of the Paradox: Philosophy and the Labyrinths of the Mind*. There is also a surprisingly large amount of material on the web. The following webpage lists a whole range of paradox sites, of very diverse kinds: www.google.com/Top/Society/Philosophy/Philosophy_of_Logic/Paradoxes/.

# 1 Zeno's paradoxes: space, time, and motion

## 1.1 Introduction

Zeno the Greek lived in Elea (a town in what is now southern Italy) in the fifth century BC. The paradox for which he is best known today concerns the great warrior Achilles and a previously unknown tortoise. For some reason now lost in the folds of time, a race was arranged between them. Since Achilles could run much faster than the tortoise, the tortoise was given a head start. Zeno's astonishing contribution is a "proof" that Achilles could never catch up with the tortoise no matter how fast he ran and no matter how long the race went on.

The supposed proof goes like this. The first thing Achilles has to do is to get to the place from which the tortoise started. The tortoise, although slow, is unflagging: while Achilles is occupied in making up his handicap, the tortoise advances a little bit further. So the next thing Achilles has to do is to get to the *new* place the tortoise occupies. While he is doing this, the tortoise will have gone on a little bit further still. However small the gap that remains, it will take Achilles some time to cross it, and in that time the tortoise will have created another gap. So however fast Achilles runs, all the tortoise need do in order not to be beaten is keep going – to make *some* progress in the time it takes Achilles to close the previous gap between them.

No one nowadays would dream of accepting the conclusion that Achilles cannot catch the tortoise. (I will not vouch for Zeno's reaction to his paradox: sometimes he is reported as having taken his paradoxical conclusions quite seriously and literally, showing that motion was impossible.) Therefore, there must be something wrong with the argument. Saying exactly *what* is wrong is not easy, and there is no uncontroversial diagnosis. Some have seen the paradox as produced by the assumption that space or time is infinitely divisible, and thus as genuinely proving that space or time is *not* infinitely divisible. Others have seen in the argument nothing more than a display of ignorance of elementary mathematics – an ignorance perhaps excusable in Zeno's time but inexcusable today.

The paradox of Achilles and the tortoise is Zeno's most famous, but there were several others. The Achilles paradox takes for granted that Achilles can start running, and purports to prove that he cannot get as far as we all know he can. This paradox dovetails nicely with one known as the Racetrack, or Dichotomy, which purports to show that nothing can *begin* to move. In order to get anywhere, say to a point one foot ahead of you, you must first get halfway there. To get to the halfway point, you must first get halfway to *that* point. In short, in order to get anywhere, even to begin to move, you must first perform an infinity of other movements. Since this seems impossible, it seems impossible that anything should move at all.

Almost none of Zeno's work survives as such. For the most part, our knowledge of what his arguments were is derived from reports by other philosophers, notably Aristotle. He presents Zeno's arguments very briefly, no doubt in the expectation that they would be familiar to his audience from the oral tradition that was perhaps his own only source. Aristotle's accounts are so compressed that only by guesswork can one reconstruct a detailed argument. The upshot is that there is no universal agreement about what should count as "Zeno's paradoxes," or about exactly what his arguments were. I shall select arguments that I believe to be interesting and important, and which are commonly attributed to Zeno, but I make no claim to be expounding what the real, historical Zeno actually said or thought.

Aristotle is an example of a great thinker who believed that Zeno was to be taken seriously and not dismissed as a mere propounder of childish riddles. By contrast, Charles Peirce wrote of the Achilles paradox: "this ridiculous little catch presents no difficulty at all to a mind adequately trained in mathematics and in logic, but is one of those which is very apt to excite minds of a certain class to an obstinate determination to believe a given proposition" (1935, vol. VI, §177, p. 122). On balance, history has sided with Aristotle, whose view on this point has been shared by thinkers as dissimilar as Hegel and Russell.

I shall discuss three Zenonian paradoxes concerning motion: the Racetrack, the Achilles, and a paradox known as the Arrow. Before doing so, however, it will be useful to consider yet another of Zeno's paradoxes, one that concerns space. Sorting out this paradox provides the groundwork for tackling the paradoxes of motion.

## 1.2 Space

In ancient times, a frequently discussed perplexity was how something ("one and the same thing") could be both one and many. For example, a book is one but also many (words or pages); likewise, a tree is one but also many (leaves, branches, molecules, or whatever). Nowadays, this is unlikely to strike anyone as very problematic. When we say that the book or the tree *is* many things, we do not mean that it is identical with many things (which would be absurd), but rather that it is made up of many parts. Furthermore, at least on the face of it, there is nothing especially problematic about this relationship between a whole and the parts which compose it (see question 1.1).

**1.1**

Appearances may deceive. Let us call some particular tree *T*, and the collection of its parts at a particular moment *P*. Since trees can survive the loss of some of their parts (e.g. their leaves in the fall), *T* can exist when *P* no longer does. Does this mean that *T* is something other than *P* or, more generally, that each thing is distinct from the sum of its parts? Can *P* exist when *T* does not (e.g. if the parts of the tree are dispersed by timber-felling operations)?

Zeno, like his teacher Parmenides, wished to argue that in such cases there are not many things but only one thing. I shall examine one ingredient of this argument. Consider any region of space, for example the region occupied by this book. The region can be thought of as having parts which are themselves spatial, that is, they have some size. This holds however small we make the parts. Hence, the argument runs, no region of space is "infinitely divisible" in the sense of containing an *infinite* number of spatial parts. For each part has a size, and a region composed of an infinite number of parts of this size must be infinite in size.

This argument played the following role in Zeno's attempt to show that it is not the case that there are "many things." He was talking only of objects in space, and he assumed that an object has a part corresponding to every part of the space it fills. He claimed to show that, if you allow that objects have parts at all, you must say that each object is infinitely large, which is absurd. You must therefore deny that objects have parts. From this Zeno went on to argue that *plurality* – the existence of many things – was impossible. I shall not consider this further development, but will instead return to the argument against infinite divisibility upon which it draws (see question 1.2).

**1.2**

* Given as a premise that no object has parts, how could one attempt to argue that there is no more than one object?

The conclusion may seem surprising. Surely one could convince oneself that any space has infinitely many spatial parts. Suppose we take a rectangle and bisect it vertically to give two further rectangles. Taking the right-hand one, bisect it vertically to give two more new rectangles. Cannot this process of bisection go on indefinitely, at least in theory? If so, any spatial area is made up of infinitely many others.

Wait one moment! Suppose that I am drawing the bisections with a ruler and pencil. However thin the pencil, the time will fairly soon come when, instead of producing fresh rectangles, the new lines will fuse into a smudge. Alternatively, suppose that I am cutting the rectangles from paper with scissors. Again, the time will fairly soon come when my strip of paper will be too small to cut. More scientifically, such a process of physical division must presumably come to an end *sometime*: at the very latest, when the remainder of the object is no wider than an atom (proton, hadron, quark, or whatever).

The proponent of infinite divisibility must claim to have no such physical process in mind, but rather to be presenting a purely intellectual process: for every rectangle we can consider, we can also consider a smaller one having half the width. This is how we conceive any space, regardless of its shape. What we have to discuss, therefore, is whether the earlier argument demonstrates that space cannot be as we tend to conceive it; whether, that is, the earlier argument succeeded in showing that no region could have infinitely many parts.

We all know that there are finite spaces which have spatial parts, but the argument supposedly shows that there are not. Therefore we must reject one of the premises that leads to this absurd conclusion, and the most suitable for rejection, because it is the most controversial, is that space is infinitely divisible. This premise supposedly forces us to say that either the parts of a supposedly infinitely divisible space are finite in size, or they are not. If the latter holds, then they are nothing, and no number of them could together compose a finite space. If the former holds, infinitely many of them together will compose an infinitely large space. Either way, on the supposition that space is infinitely divisible, there are no finite spaces. Since there obviously are finite spaces, the supposition must be rejected.

The notion of infinite divisibility remains ambiguous. On the one hand, to say that any space is infinitely divisible could mean that there is no upper limit to the number of imaginary operations of dividing we could effect. On the other hand, it could mean that the space contains an infinite number of parts. It is not obvious that the latter follows from the former. The latter claim might seem to rely on the idea that the process of imaginary dividings could somehow be "completed." For the moment let us assume that the thesis of infinite divisibility at stake is the thesis that space contains infinitely many non-overlapping parts, and that each part has some finite size.

The most doubtful part of the argument against the thesis is the claim that a space composed of an infinity of parts, each finite in size, must be infinite. This claim is incorrect, and one way to show it is to appeal to mathematics. Let us represent the imagined successive bisections by the following series:

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8} \ldots$$

where the first term $\left(\frac{1}{2}\right)$ represents the fact that, after the first bisection, the right-hand rectangle is only half the area of the original rectangle; and similarly for the other terms. Every member of this series is a finite number, just as each of the spatial parts is of finite size. This does not mean that the sum of the series is infinite. On the contrary, mathematics texts have it that this series sums to 1. If we find nothing problematic in the idea that an infinite collection of finite numbers has a finite sum, then by analogy we should be happy with the idea that an infinite collection of finite spatial parts can compose a finite spatial region (see question 1.3).

This argument from mathematics establishes the analogous point about space (namely, that infinitely many parts of finite size may together form a finite whole) only upon the assumption that the analogy is good: that space, in the respect in question, has the properties that numbers have. This is controversial. For example, we have already said that some people take Zeno's paradoxes to show that space is not continuous, although the series of numbers is. Hence we would do well to approach the issue again. We do not have to rely on any mathematical argument to show that a finite whole can be composed of an infinite number of finite parts.

There are two rather similar propositions, one true and one false, and we must be careful not to confuse them.

(1) If, for some finite size, a whole contains infinitely many parts none smaller than this size, then the whole is infinitely large.
(2) If a whole contains infinitely many parts, each of some finite size, then the whole is infinitely large.

Statement (1) is true. To see this, let the minimum size of the parts be $\delta$ (say linear or square or cubic inches). Then the size of the whole is $\infty \times \delta$, which is clearly an infinite number. However, (1) does not bear on the case we are considering. To see this, let us revert to our imagined bisections. The idea was that however small the remaining area was, we could

**1.3**

Someone might object: is it not just a *convention* in mathematics to treat this series as summing to 1? More generally, is it not just a convention to treat the sum of an infinite series as the limit of the partial sums? If this is a mere mathematical convention, how can it tell us anything about space? Readers with mathematical backgrounds might like to comment on the following argument, which purports to show that the fact that the series sums to 1 can be derived from ordinary arithmetical notions, without appeal to any special convention. (*Warning*: mathematicians tell me that what follows is highly suspect!)

The series can be represented as

$$x + x^2 + x^3 + \cdots$$

where $x = ½$. Multiplying this expression by $x$ has the effect of lopping off the first term:

$$x(x + x^2 + x^3 + \cdots) = x^2 + x^3 + x^4 + \cdots$$

Here we apply a generalization of the principle of distribution:

$$a.(b + c) = (a.b) + (a.c).$$

Using this together with a similar generalization of the principle that

$$(1 - a).(b + c) = (b + c) - a.(b + c)$$

we get:

$$(1 - x).(x + x^2 + x^3 + \cdots) = (x + x^2 + x^3 + \cdots) - (x^2 + x^3 + x^4 + \cdots)$$

Thus

$$(1 - x).(x + x^2 + x^3 + x \ldots) = x$$

So, dividing both sides by $(1 - x)$:

$$x + x^2 + x^3 + \cdots = \frac{x}{(1 - x)}$$

So where $x = ½$, the sum of the series is equal to 1.

always imagine it being divided into two. This means that there can be no lower limit on how small the parts are. There can be no size δ such that all the parts are at least this big. For any such size, we can always imagine it being divided into two.

To see that (2) is false, we need to remember that it is essential to the idea of infinite divisibility that the parts get smaller, without limit, as the imagined process of division proceeds. This gives us an almost visual way of understanding how the endless series of rectangles can fit into the original rectangle: by getting progressively smaller.

It would be as wrong to infer "There is a finite size which every part possesses" from "Every part has some finite size or other" as it would be to infer "There is a woman who is loved by every man" from "Every man loves some woman or other." (Readers trained in formal logic will recognize a quantifier-shift fallacy here: one cannot infer an ∃∀ conclusion from the corresponding ∀∃ premise.)

The explanation for any tendency to believe that (2) is true lies in a tendency to confuse it with (1). We perhaps tend to think: *at the end of the series* the *last* pair of rectangles formed have some finite size, and all the other infinitely many rectangles are larger. Therefore, taken together they must make up an infinite area. However, there is *no such thing* as the last pair of rectangles to be formed: our infinite series of divisions has no last member. Once we hold clearly in mind that there can be no lower limit on the size of the parts induced by the infinite series of envisaged divisions, there is no inclination to suppose that having infinitely many parts entails being infinitely large.

The upshot is that there is no contradiction in the idea that space is infinitely divisible, in the sense of being composed of infinitely many non-overlapping spatial parts, each of some finite (non-zero) size. This does not establish that space *is* infinitely divisible. Perhaps it is granular, in the way in which, according to quantum theory, energy is. Perhaps there are small spatial regions that have no distinct subregions. The present point, however, is that the Zenonian argument we have discussed gives us no reason at all to believe this granular hypothesis.

This supposed paradox about space may well not strike us as very deep, especially if we have some familiarity with the currently orthodox mathematical treatment of infinity. Still, we must not forget that current orthodoxy was not developed without a struggle, and was achieved several centuries after Zeno had pondered these questions. Zeno and his contemporaries might with good reason have had more trouble with it than we do. The position of a paradox on the ten-point scale mentioned in the introduction can change over time: as we become more sophisticated detectors of mere appearance, a paradox can slide down toward the Barber end of the scale.

Clearing this paradox out of the way will prove to have been an essential preliminary to discussing Zeno's deeper paradoxes, which concern motion.